



PHYS 102 – General Physics II Final Exam

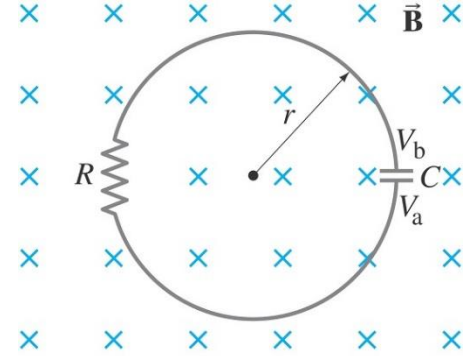
Duration: 120 minutes

Monday, 20 May 2024; 18:00

1. A circular-shaped circuit of radius r , containing a resistance R and capacitance C , is situated with its plane perpendicular to a spatially uniform magnetic field $\vec{B}$ directed into the page. Starting at time $t = 0$, the voltage difference $V_{ba} = V_b - V_a$ across the capacitor plates is observed to increase with time t as $V_{ba} = V_0(1 - e^{-t/\tau})$, where V_0 and $\tau = RC$ are positive constants.

(a) (10 Pts.) Determine the potential drop across the resistor as a function of time.

(b) (15 Pts.) Determine dB/dt , the rate at which the magnetic field magnitude changes with time. Is B becoming larger or smaller as time increases?



Solution:

(a) Current through the circuit is equal to the current through the capacitor.

$$I = \frac{dQ}{dt} = \frac{d}{dt}(CV_{ba}) = C \frac{dV_{ba}}{dt} \rightarrow I = CV_0 \frac{d}{dt}(1 - e^{-t/\tau}) = \frac{V_0}{R} e^{-t/RC}.$$

Hence, the potential drop across the resistor is

$$V_R = IR \rightarrow V_R = V_0 e^{-t/RC}.$$

(b) Flux of the magnetic field through the circular circuit is

$$\Phi_B = \pi r^2 B \rightarrow |\mathcal{E}_{\text{ind}}| = \left| \frac{d\Phi_B}{dt} \right| = \pi r^2 \left| \frac{dB}{dt} \right|.$$

Applying Kirchoff's loop rule around the circuit, we have

$$\mathcal{E}_{\text{ind}} = V_{ba} + V_R \rightarrow \pi r^2 \left| \frac{dB}{dt} \right| = V_0(1 - e^{-t/RC}) + V_0 e^{-t/RC} = V_0.$$

Therefore,

$$\left| \frac{dB}{dt} \right| = \frac{V_0}{\pi r^2}.$$

Since V_{ba} is increasing, charge is building up on the top plate of the capacitor meaning that the current is clockwise.

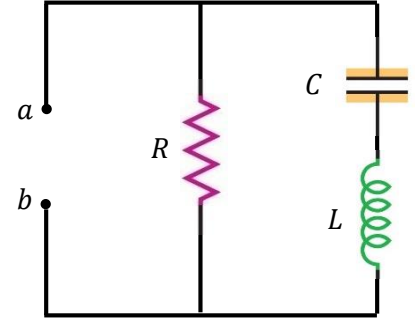
By Lenz's law, the current should be opposing the change, meaning that the magnetic field must be **decreasing**.

2. Consider the circuit in the figure. At time $t = 0$, a DC source with emf $\mathcal{E}$ is connected to points a and b .

(a) (5 Pts.) Find currents through each element (R , C , and L) at time $t = 0$.

(b) (5 Pts.) Find the final charge on the capacitor.

(c) (15 Pts.) Instead of a DC source, an AC source $V = V_0 \cos \omega t$ is connected to points a and b . What is the amplitude of the AC current supplied by the source?



Solution: (a) With $V_{ba} = \mathcal{E}$ at time $t = 0$

$$I_R = \frac{\mathcal{E}}{R}, \quad I_C = I_L = 0.$$

(b) With the capacitor fully charged, we again have $I_R = \mathcal{E}/R$, $I_C = I_L = 0$. Since in this case $V_C = V_R = \mathcal{E}$, we have

$$\frac{Q}{C} = \mathcal{E} \rightarrow Q = \mathcal{E}C.$$

(c) In this case $V_{ba} = V_0 \cos \omega t$, and i_R is in phase with V_{ba} , and we have

$$I_R = \frac{V_0}{R}.$$

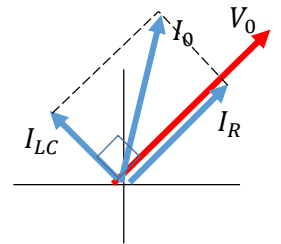
Since $I_C = I_L$ (connected in series), the phasor diagram for the LC branch is shown.

Total impedance of the LC branch is

$$Z_{LC} = \left(\omega L - \frac{1}{\omega C} \right) = \frac{V_0}{I_{LC}} \rightarrow I_{LC} = \frac{V_0}{\left(\omega L - \frac{1}{\omega C} \right)}.$$

Since $I_0 = \sqrt{I_{LC}^2 + I_R^2}$, we have

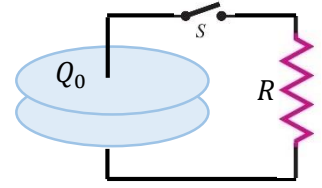
$$I_0 = V_0 \sqrt{\frac{1}{R^2} + \frac{1}{\left(\omega L - \frac{1}{\omega C} \right)^2}}.$$



3. A parallel plate capacitor is built from two metal discs of radius a separated by a distance d . Initially this capacitor is charged with total charge Q_0 . At time $t = 0$, a switch is closed and the capacitor starts discharging over a resistor with resistance R .

(a) (10 Pts.) Find the magnitude of the electric field (assumed uniform) inside the capacitor as a function of time.

(b) (15 Pts.) Find the magnitude of the magnetic field at a perpendicular distance r from the symmetry axis of the capacitor as a function of time.



Solution:

(a) For the parallel plate capacitor we have $C = \epsilon_0 \pi a^2 / d$. When the switch is closed, the capacitor will discharge over the resistance and the charge at any time on the capacitor will be

$$q(t) = Q_0 e^{-t/RC}.$$

The magnitude of the electric field inside the capacitor is $E = \sigma / \epsilon_0$. Since $\sigma = q / (\pi a^2)$, we have

$$E = \frac{Q_0}{\epsilon_0 \pi a^2} e^{-t/RC}.$$

(b) Changing electric field will create a magnetic field according to Ampère-Maxwell law.

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}.$$

Integrating over a circle of radius r centered at the symmetry axis, above equation becomes

$$2\pi r B(r) = \mu_0 \epsilon_0 \pi r^2 \left| \frac{dE}{dt} \right| \rightarrow B(r) = \frac{\mu_0 Q_0 r}{2\pi a^2 RC} e^{-t/RC}.$$

Since $C = \epsilon_0 \pi a^2 / d$, we get

$$B(r) = \frac{\mu_0 Q_0 r d}{2\epsilon_0 (\pi a^2)^2 R} e^{-t/RC}.$$

4. The superposition of two electromagnetic plane waves with the same frequency in free space creates the following electric field:

$$\vec{E}(\vec{r}, t) = E_0 \hat{i} \cos\left(\frac{2\pi z}{z_0}\right) \cos(k_0 y - \omega t),$$

where $\omega = 6\pi \times 10^{10}$ rad/s, $k_0 = \sqrt{3}\pi \times 10^2$ m⁻¹, $E_0 = 3$ N/C and the speed of light in vacuum is $c = 3 \times 10^8$ m/s. In your answers DO NOT substitute π numerically, leave it as π .

- (a) (5 Pts.) What is the frequency of the plane waves?
- (b) (5 Pts.) What is the speed of the above wave in the $\hat{j}$ direction?
- (c) (5 Pts.) Find the wave vectors of the two plane electromagnetic waves.
- (d) (5 Pts.) Calculate the value of z_0 .
- (e) (5 Pts.) Calculate the magnitude of the magnetic field vector at $\vec{r} = 0$ at time $t = 0$.

Solution: (a)

$$f = \frac{\omega}{2\pi} \rightarrow f = 3 \times 10^{10} \text{ Hz}.$$

(b)

$$v_y = \frac{\omega}{k_0} \rightarrow v_y = 2\sqrt{3} \times 10^8 \text{ m/s}.$$

(c) Using the trigonometric identity $\cos a \cos b = \frac{1}{2} \cos(a+b) + \frac{1}{2} \cos(a-b)$, we get

$$\vec{E}(\vec{r}, t) = \frac{1}{2} E_0 \hat{i} \left[\cos\left(k_0 y + \frac{2\pi z}{z_0} - \omega t\right) + \cos\left(k_0 y - \frac{2\pi z}{z_0} - \omega t\right) \right].$$

This is a superposition of two plane waves with direction of propagation

$$\vec{k}_1 = k_0 \hat{j} + \frac{2\pi}{z_0} \hat{k}, \text{ and } \vec{k}_2 = k_0 \hat{j} - \frac{2\pi}{z_0} \hat{k}, \text{ respectively. Since } k_1 = k_2 = \sqrt{k_0^2 + \frac{4\pi^2}{z_0^2}},$$

and both waves satisfy $ck_{1,2} = \omega$, we have

(d) Since $c = \omega/k$, we have

$$c \sqrt{k_0^2 + \frac{4\pi^2}{z_0^2}} = \omega \rightarrow z_0 = \frac{2\pi}{\sqrt{\left(\frac{\omega}{c}\right)^2 - k_0^2}} = 2 \times 10^{-2} \text{ m} = 2 \text{ cm}.$$

(e) Two interfering waves have the same electric field component but their directions of propagation are different. Therefore the two waves will have different magnetic field components with equal magnitudes $B_1 = B_2 = E_0/2c$, and the total magnetic field is the vector sum of the two. Noting that $\vec{E} \times \vec{B}_1$ is in the direction of $\vec{k}_1$, and $\vec{E} \times \vec{B}_2$ is in the direction of $\vec{k}_2$, and using the figure given, we have

$$B = 2 \left(\frac{E_0}{2c} \right) \cos \theta.$$

Since $\cos \theta = ck_0/\omega = \sqrt{3}/2$, we find

$$B = 2 \left(\frac{E_0}{2c} \right) \cos \theta = \frac{\sqrt{3}}{2} \times 10^{-8} \text{ T}.$$

